A New Modification of Shukla Distribution
With Properties and Lifetime Data

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Abstract - In this paper, we have introduced a new model of Shukla distribution using weighted technique known as Weighted Shukla distribution by considering the weight function as \( w(x) = x^c \) where the weight parameter is \( c \). The different characteristics as well as the structural properties of the proposed weighted Shukla distribution have also been derived and discussed. The Parameters of the newly proposed model have also been estimated by using the method of maximum likelihood estimation and also the Fisher’s information matrix has been discussed. Finally the newly proposed model has also been demonstrated with two real lifetime data sets for examining the suitability of the model.

KeyWords - Weighted distribution, Shukla distribution, Reliability analysis, Order statistics, Entropies, Maximum likelihood estimation.

I. INTRODUCTION

The Recently developed new model of probability distribution known as Shukla distribution and its Applications was introduced by Shukla et al. (2019) is a newly proposed two parametric lifetime model. The several one parametric lifetime distributions including exponential, Shanker, Pranav, ishita and Ram Awadh are particular cases of it. The newly introduced probability model known as Shukla distribution can also be applied for generating several one parametric lifetime distributions. Shukla has also obtained its various mathematical and statistical properties including its hazard rate function, mean residual life function and stochastic ordering. The parameters of the newly proposed model has also been estimated by using the method of maximum likelihood estimation. Shukla distribution has also been demonstrated with real lifetime data sets and it is found that the newly introduced two parametric Shukla distribution has better flexibility in handling real lifetime data over one and two parametric lifetime distributions.

The probability density function of Shukla distribution (SD) is given by

\[
f(x; \theta, \alpha) = \frac{\theta^{\alpha + 1}}{\theta^{\alpha + 1} + \Gamma(\alpha + 1)} \left( \frac{\theta}{\theta + x} \right)^{\alpha - 1} e^{-\theta x}; \quad x > 0, \theta > 0, \alpha \geq 0
\]  

(1)

and the cumulative distribution function of the Shukla distribution is given by
The concept of weighted probability distribution is applied in various research areas related to reliability, biomedicine, ecology and branching process. Fisher (1934) introduced the concept of weighted distributions for the purpose of modelling the ascertainment bias. While for modelling the statistical data, Rao (1965) developed this concept in an integrated manner when the standard distributions were not appropriate to record the observations with equal probabilities. Warren (1975) was the first to apply the weighted distributions in connection with sampling wood cells. Patil and Rao (1978) introduced the concept of size biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. Weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded, instead they are recorded according to some weight function. The study of weighted distributions is useful in the theory of distributions, because it provides a new understanding of the existing standard distributions. There are various good sources which provide detailed description of weighted distributions. Different authors have studied and reviewed the various weighted probability models and illustrated their application in different fields. For survival data analysis, Jing (2010) have introduced the weighted inverse weibull distribution and beta-inverse weibull distribution as a new lifetime model. Subramanian and Rather (2018) have obtained the weighted version of exponentiated mukherjee-Islam distribution. Hasan et al. (2018) discussed on three parameters weighted quasi Akash distribution with properties and Applications. Mudasir and Ahmad have obtained the weighted version of Erlang distribution with characterizations and information measures. Rather and Subramanian (2019) discussed On weighted sushila distribution with properties and Applications. Recently Ganaie and Rajagopalan (2020) discussed the weighted version of quasi gamma distribution with properties and applications.

Suppose \( X \) is a random variable of non-negative sign with probability density function \( f(x) \). Let \( w(x) \) be the non-negative weight function, then, the probability density function of the weighted random variable \( X_w \) is given by

\[
f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad x > 0.
\]

Where \( w(x) \) be the non-negative weight function and \( E(w(x)) = \int w(x)f(x)dx < \infty \).

Weighted models are of various choices, when \( w(x) = x^c \), the resulting distribution is known as weighted distribution. The weighted model of Shukla distribution is obtained by applying the weight function \( w(x) = x^c \) to Shukla distribution in order to obtain the weighted shukla distribution. The probability density function of weighted shukla distribution is given by

\[
F(x; \theta, \alpha) = 1 - \frac{\theta^\alpha (\theta + x)^\alpha e^{-\theta x} + \alpha \Gamma(\alpha, \theta x)}{\theta^\alpha + 1 + \Gamma(\alpha + 1)}; \quad x > 0, \theta > 0, \alpha \geq 0
\]
The corresponding cumulative distribution function of weighted Shukla distribution is obtained as

\[
F_w(x; \theta, \alpha, c) = \frac{1}{\theta^{\alpha+1} c! (\alpha+c)!} \int_0^x \left( \frac{\theta^{\alpha+1} + \alpha c + 1}{\theta^{\alpha+1} c! (\alpha+c)!} e^{-\theta x} \right) dx
\]

(4)

Put \( \theta x = t \Rightarrow dt = dx \Rightarrow dx = \frac{dt}{\theta} \). When \( x \to x, t \to \theta x \) and as \( x \to 0, t \to 0 \).

After simplifying equation (4), we obtain cumulative distribution function of weighted Shukla distribution

\[
F_w(x; \theta, \alpha, c) = \frac{1}{\theta^{\alpha+1} c! (\alpha+c)!} \left( \theta^{\alpha+1} \gamma(c+1, \theta x) + \gamma(\alpha+c+1, \theta x) \right)
\]

(5)

Where \( c, \theta \) and \( \alpha \) are positive parameters and \( \gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt \) is a lower incomplete gamma function.
In this section, we have obtained the Reliability, hazard rate and Reverse hazard rate function of the Proposed weighted shukla distribution.

2.1 Reliability function

The reliability function is also known as survival function. it can be computed as complement of the cumulative distribution function of the model. The reliability function or the survival function of weighted Shukla distribution can be computed as

\[ R(x) = 1 - F_w(x; \theta, \alpha, c) \]

\[ R(x) = 1 - \frac{1}{\theta^{\alpha+1} e^{\gamma(c+1, \theta x)} + \gamma(\alpha + c + 1, \theta x)} \]

2.2 Hazard function

The hazard function is also known as instantaneous failure rate or force of mortality and is given by

\[ h(x) = \frac{f_w(x; \theta, \alpha, c)}{R(x)} \]

\[ h(x) = \frac{x^c \theta^{\alpha+c+1} e^{-\theta x}}{\left( \theta^{\alpha+1} e^{(c+1, \theta x)} + \gamma(\alpha + c + 1, \theta x) \right) \left( \theta + x^\alpha \right) e^{-\theta x}} \]

2.3 Reverse hazard function

The reverse hazard function of weighted shukla distribution is given by
\[ h^r(x) = \frac{F_W(x; \theta, \alpha, c)}{F_W(x; \theta, \alpha, c)} \]

\[ h^r(x) = \frac{x^c \theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) e^{-\theta x} \]

IV. STATISTICAL MEASURES

In this Portion, we have obtain the different statistical properties of weighted Shukla distribution including its moments, Harmonic mean, moment generating function and characteristics function

4.1 Moments

Let \(X\) denotes the random variable of weighted Shukla distribution with parameters \(\theta, \alpha\) and \(c\) then the \(r^{th}\) order moment \(E(X^r)\) of Weighted Shukla distribution is given by

\[ E(X^r) = \mu_r^* = \int_0^\infty x^r f_w(x; \alpha, \theta, c) dx \]

\[ E(X^r) = \mu_r^* = \int_0^\infty x^r \frac{x^c \theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) e^{-\theta x} dx \]

\[ = \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) \int_0^\infty x^c e^{-\theta x} dx \]

\[ = \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) \theta^c \frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)} \]

\[ = \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) \theta^c \frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)} \]

\[ = \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) \theta^c \frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)} \]

\[ = \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1} \frac{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)}{\gamma(c+1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left(\frac{\theta^\alpha}{\theta^\alpha + 1} \right) \theta^c \frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)} \]
\[ E(X^r) = \mu_r^* = \frac{\theta^{\alpha+1} (c+r)! + (\alpha + c + r)!}{\theta^r (\theta^{\alpha+1} c! + (\alpha + c)!)} \]  

Substitute \( r = 1, 2, 3 \) and \( 4 \) in equation (6) we get moments of weighted Shukla distribution.

\[ E(X) = \mu_1^* = \frac{\theta^{\alpha+1} (c+1)! + (\alpha + c + 1)!}{\theta (\theta^{\alpha+1} c! + (\alpha + c)!)}, \]

\[ E(X^2) = \mu_2^* = \frac{\theta^{\alpha+1} (c+2)! + (\alpha + c + 2)!}{\theta^2 (\theta^{\alpha+1} c! + (\alpha + c)!)}, \]

\[ E(X^3) = \mu_3^* = \frac{\theta^{\alpha+1} (c+3)! + (\alpha + c + 3)!}{\theta^3 (\theta^{\alpha+1} c! + (\alpha + c)!)}, \]

\[ E(X^4) = \mu_4^* = \frac{\theta^{\alpha+1} (c+4)! + (\alpha + c + 4)!}{\theta^4 (\theta^{\alpha+1} c! + (\alpha + c)!)}. \]

Variance \((\mu_2)\) = \[ \frac{\theta^{\alpha+1} (c+2)! + (\alpha + c + 2)!}{\theta^2 (\theta^{\alpha+1} c! + (\alpha + c)!)^2} - \left( \frac{\theta^{\alpha+1} (c+1)! + (\alpha + c + 1)!}{\theta (\theta^{\alpha+1} c! + (\alpha + c)!)} \right)^2 \]

\[ S.D(\sigma) = \sqrt{\frac{\theta^{\alpha+1} (c+2)! + (\alpha + c + 2)!}{\theta^2 (\theta^{\alpha+1} c! + (\alpha + c)!)^2} - \left( \frac{\theta^{\alpha+1} (c+1)! + (\alpha + c + 1)!}{\theta (\theta^{\alpha+1} c! + (\alpha + c)!)} \right)^2} \]

**4.2 Harmonic mean**

The Harmonic mean is the reciprocal of the arithmetic mean of the reciprocals. The harmonic mean for the proposed weighted Shukla distribution can be obtained as

\[ H.M = E\left(\frac{1}{x}\right) = \int_{0}^{\infty} \frac{1}{x} f_w(x; \theta, \alpha, c) dx \]
\[
\int_{0}^{\infty} x^{c-1} \theta^{\alpha + c + 1} e^{-\theta x} dx
\]

\[
= \frac{\theta^{\alpha + c + 1}}{\theta^{\alpha + 1} c! + (\alpha + c)!} \left( \int_{0}^{\infty} x^{c-1} e^{-\theta x} dx + \int_{0}^{\infty} x^{c-1} e^{-\theta x} dx \right)
\]

\[
= \frac{\theta^{\alpha + c + 1}}{\theta^{\alpha + 1} c! + (\alpha + c)!} \left( \int_{0}^{\infty} x^{c-1} e^{-\theta x} dx + \int_{0}^{\infty} x^{c-1} e^{-\theta x} dx \right)
\]

\[
\Rightarrow H.M = \frac{\theta^{\alpha + c + 1}}{\theta^{\alpha + 1} c! + (\alpha + c)!} \left( \theta \gamma(c + 1, \theta) + \gamma(\alpha + c + 1, \theta) \right)
\]

4.3 Moment Generating Function

The moment generating function is the expectation of a function of the random variable. we begin with the well-known definition of the moment generating function given by

\[
M_X(t) = E(e^{tx}) = \int_{0}^{\infty} e^{tx} f_X(x; \theta, \alpha, c) dx
\]

\[
= \int_{0}^{\infty} \left(1 + tx + \frac{(tx)^2}{2!} + \cdots \right) f_X(x; \theta, \alpha, c) dx
\]

\[
= \int_{0}^{\infty} \sum_{j=0}^{\infty} \frac{t^j x^j}{j!} f_X(x; \theta, \alpha, c) dx
\]

\[
= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu^j
\]

\[
\Rightarrow M_X(t) = \frac{1}{\theta^{\alpha + 1} c! + (\alpha + c)!} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\theta^{\alpha + 1} (c + j)! + (\alpha + c + j)! \right)
\]
4.4 Characteristic function

In probability theory and statistics characteristic function is defined as the function of any real-valued random variable completely defines the probability distribution of a random variable. The characteristic function of the proposed weighted Shukla distribution is given by

\[ \varphi_X(it) = M_X(it) \]

\[ M_X(it) = \frac{1}{\theta^{\alpha + 1}c! + (\alpha + c)!} \sum_{j=0}^{\infty} \frac{it^j}{j!\theta^j} \left( \theta^\alpha + 1 \right)^{(c + j)! + (\alpha + c + j)!} \]

V. ORDER STATISTICS

Order Statistics are the sequence of samples arranged in an ascending order. Let \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) denotes the order statistics of a random sample \( X_1, X_2, \ldots, X_n \) from a Continuous distribution with cumulative distribution function \( F_X(x) \) and probability density function \( f_X(x) \), then the probability density function of \( r^{th} \) order statistics \( X_{(r)} \) is given by

\[ f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) (F_X(x))^{r-1} \left( 1 - F_X(x) \right)^{n-r} \]

Using equation (3) and (5) the probability density function of \( r^{th} \) order statistics of weighted Shukla distribution is given by

\[ f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \left( \frac{x^c \theta^{\alpha + c + 1}}{\theta^{\alpha + 1}c! + (\alpha + c)!} \right) \left( \theta^\alpha + 1 \right)^{\gamma(c + 1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left( 1 - \frac{1}{\theta^{\alpha + 1}c! + (\alpha + c)!} \right)^{r-1} \left( 1 - \frac{1}{\theta^{\alpha + 1}c! + (\alpha + c)!} \right)^{n-r} \]

Therefore, the probability density function of \( i^{th} \) order statistics \( X_{(i)} \) of Weighted Shukla distribution is given by

\[ f_{X_{(i)}}(x) = \frac{nx^c \theta^{\alpha + c + 1}}{\theta^{\alpha + 1}c! + (\alpha + c)!} \left( \theta^\alpha + 1 \right)^{\gamma(c + 1, \theta x) + \gamma(\alpha + c + 1, \theta x)} \left( 1 - \frac{1}{\theta^{\alpha + 1}c! + (\alpha + c)!} \right)^{i-1} \left( 1 - \frac{1}{\theta^{\alpha + 1}c! + (\alpha + c)!} \right)^{n-i} \]

and the Probability density function of higher order statistics \( X_{(n)} \) of Weighted Shukla distribution is given by
VI. LIKELIHOOD RATIO TEST

Suppose the random samples $X_1, X_2, \ldots, X_n$ of size $n$ drawn from the Shukla distribution or weighted Shukla distribution. We set up the hypothesis

$$H_0 : f(x) = f(x; \theta, \alpha)$$

against

$$H_1 : f(x) = f_w(x; \theta, \alpha, c)$$

Whether the random sample of size $n$ comes from Shukla distribution or Weighted Shukla distribution, the following test statistic procedure is used

$$\Delta = \frac{L_1}{L_0} = \frac{n}{\prod_{i=1}^{n} f_w(x; \theta, \alpha, c)} f(x; \theta, \alpha)$$

$$\Delta = \frac{L_1}{L_0} = \frac{n}{\prod_{i=1}^{n} \frac{x_i^c \theta^c (\theta^\alpha + \alpha)!}{\theta^\alpha + 1 c! + (\alpha + c)!}}$$

$$\Delta = \frac{L_1}{L_0} = \left( \frac{\theta^c (\theta^\alpha + \alpha)!}{\theta^\alpha + 1 c! + (\alpha + c)!} \right)^n \prod_{i=1}^{n} x_i^c$$

We reject the null hypothesis if

$$\Delta = \left( \frac{\theta^c (\theta^\alpha + \alpha)!}{\theta^\alpha + 1 c! + (\alpha + c)!} \right)^n \prod_{i=1}^{n} x_i^c > k$$

Equivalently, we reject the null hypothesis if

$$\Delta^* = \prod_{i=1}^{n} x_i^c > k^*, \text{ where } k^* = k \left( \frac{\theta^c (\theta^\alpha + \alpha)!}{\theta^\alpha + 1 c! + (\alpha + c)!} \right)^n$$
When the sample of size \( n \) is large, \( 2\log \beta \) is distributed as chi-square distribution with one degree of freedom and also from the chi-square distribution, p-value is obtained. Also, we reject the null hypothesis, when the probability value is given by

\[
p\left( \Delta^* > \beta^* \right)
\]

Where \( \beta^* = \left( \prod_{i=1}^{n} x_i^c \right) \) is less than a particular level of significance and \( \left( \prod_{i=1}^{n} x_i^c \right) \) is the observed value of statistic \( \Delta^* \).

VII. BONFERRONI AND LORENZ CURVES

The Bonferroni and the Lorenz curves have assumed relief not only in economics to study the distribution of income and wealth or income and poverty, but also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

\[
B(p) = \frac{\int_{0}^{q} x f_w(x; \theta, \alpha, c) dx}{\mu_1^{-1}(q)}
\]

and

\[
L(p) = pB(p) = \frac{\int_{0}^{q} x f_w(x; \theta, \alpha, c) dx}{\mu_1^{-1}(q)}
\]

Where \( \mu_1' = E(X) = \frac{\theta^{\alpha+1}(c+1)! + (\alpha + c + 1)!}{\theta^c \alpha+1 c! + (\alpha + c)!} \) and \( q = F^{-1}(p) \)

\[
B(p) = \frac{\theta^{\alpha+1}(c+1)! + (\alpha + c + 1)!}{p\left( \theta^{\alpha+1}(c+1)! + (\alpha + c + 1)! \right)} \frac{q^{c+1}}{\theta^c \alpha+1 c! + (\alpha + c)!} \left( \theta + cx \right) - \theta x dx
\]

\[
B(p) = \frac{\theta^{\alpha+c+2}}{p\left( \theta^{\alpha+1}(c+1)! + (\alpha + c + 1)! \right)} \frac{q^{c+1}}{\theta^c x^{\alpha+1}} \left( \theta + cx \right) - \theta x dx
\]

\[
B(p) = \frac{\theta^{\alpha+c+2}}{p\left( \theta^{\alpha+1}(c+1)! + (\alpha + c + 1)! \right)} \left( \frac{q e^{-\theta x} x^{c+2}}{0} - x^{\alpha+c+2} + 1 dx + \frac{q e^{-\theta x} x^{c+2} - 1 dx}{0} \right)
\]

\[
B(p) = \frac{\theta^{\alpha+c+2}}{p\left( \theta^{\alpha+1}(c+1)! + (\alpha + c + 1)! \right)} \left( \theta f(c+2, \theta q) + \gamma(\alpha + c + 2, \theta q) \right)
\]

\[
L(p) = \frac{\theta^{\alpha+c+2}}{p\left( \theta^{\alpha+1}(c+1)! + (\alpha + c + 1)! \right)} \left( \theta f(c+2, \theta q) + \gamma(\alpha + c + 2, \theta q) \right)
\]
VIII. ENTROPIES

Entropies are important in different areas of knowledge such as probability, statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty.

8.1 Renyi Entropy

The Renyi entropy is an important one among the entropies, applied in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

\[ e(\beta) = \frac{1}{1 - \beta} \log \left( \int f_w(x; \theta, \alpha, c) dx \right) \]

Where, \( \beta > 0 \) and \( \beta \neq 1 \)

\[ e(\beta) = \frac{1}{1 - \beta} \log \left( \frac{c^\alpha (e^\theta + c + 1)}{c! + (\alpha + c)!} \left( \theta + x \right)^{-\alpha} e^{-\theta x} dx \right)^\beta \]

Using Binomial expansion in (7), we obtain

\[ e(\beta) = \frac{1}{1 - \beta} \log \left( \sum_{k=0}^{\infty} \left( \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1 + c!(\alpha + c)!} \right)^\beta \right) \]

\[ e(\beta) = \frac{1}{1 - \beta} \log \left( \sum_{k=0}^{\infty} \left( \frac{\theta^\alpha + c + 1}{\theta^\alpha + 1 + c!(\alpha + c)!} \right)^\beta \right) \]

8.2 Tsallis Entropy

A generalization of Boltzmann-Gibbs (B.G) statistical properties initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows
Using Binomial expansion in equation (8), we get

\[ S_\gamma = \frac{1}{\gamma - 1} \left( 1 - \left( \frac{\theta^{\alpha + c + 1}}{\theta^{\alpha + c} + (\alpha + c)!} \right)^\gamma \sum_{j=0}^{\infty} \left( \begin{array}{c} \gamma \\ j \end{array} \right) (\gamma + c + j + 1 - 1) \right) \]

IX. MAXIMUM LIKELIHOOD ESTIMATION AND FISHER’S INFORMATION MATRIX

In this portion, the method of Maximum likelihood estimate is also used for estimating the parameters of the weighted Shukla distribution. Let \( x_1, x_2, x_3, ..., x_n \) be a random sample of size \( n \) from the weighted Shukla distribution, then the corresponding likelihood function is given by

\[ L(x) = \prod_{i=1}^{n} f_w(x; \theta, \alpha, c) \]

\[ L(x) = \prod_{i=1}^{n} \frac{x_i^c \theta^{\alpha + c + 1}}{\theta^{\alpha + c} + (\alpha + c)!} (\theta + x_i^\alpha e^{-\theta x_i}) \]

\[ L(x) = \frac{\theta^n (\alpha + c + 1)}{(\theta^{\alpha + c} + (\alpha + c)!)} \prod_{i=1}^{n} x_i^c (\theta + x_i^\alpha e^{-\theta x_i}) \]

The log likelihood function is given by
\[
\log L = n(\alpha + c + 1) \log \theta - n \log \left(\frac{\theta^{\alpha+1} c^c (\alpha + c)!}{\alpha + 1 c^c (\alpha + c)!}\right) + c \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log \left(\theta + x_i^{\alpha}\right) - \theta \sum_{i=1}^{n} x_i
\]  

(9)

Differentiating the log likelihood equation (9) with respect to parameters \(\theta, \alpha\) and \(c\). We obtain the normal equations as

\[
\frac{\partial \log L}{\partial \theta} = \frac{n(\alpha + c + 1)}{\theta} - n \left(\frac{(\alpha + 1)\theta^{\alpha} c!}{\theta^{\alpha+1} c^c (\alpha + c)!}\right) + \frac{n}{\sum_{i=1}^{n} \left(\frac{1}{\theta + x_i^{\alpha}}\right)} = 0
\]

\[
\frac{\partial \log L}{\partial \alpha} = n \log \theta - n \left(\frac{\theta^{\alpha+1} c^c (\alpha + c)!}{\theta^{\alpha+1} c^c (\alpha + c)!}\right) = 0
\]

\[
\frac{\partial \log L}{\partial c} = n \log \theta - n \left(\frac{\theta^{\alpha+1} c^c (\alpha + c)!}{\theta^{\alpha+1} c^c (\alpha + c)!}\right) + \sum_{i=1}^{n} \log x_i = 0
\]

Because of the complicated form of above likelihood equations, algebraically it is too difficult to solve these non-linear system of equations. Therefore we use R and wolfram mathematics for estimating the parameters of the proposed weighted Shukla distribution.

To obtain the confidence interval, we use the asymptotic normality tests. We have that, if \(\hat{\gamma} = (\hat{\theta}, \hat{\alpha}, \hat{c})\) denotes the maximum likelihood estimate of \(\gamma = (\theta, \alpha, c)\). We can state the result as

\[
\sqrt{n}(\hat{\gamma} - \gamma) \rightarrow N_3(0, I^{-1}(\gamma))
\]

Where \(I(\gamma)\) is Fisher’s Information matrix

The elements of 3x3 Fisher’s Information matrix is given below

\[
I(\gamma) = \frac{1}{n} \begin{bmatrix}
E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) \\
E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(\frac{\partial^2 \log L}{\partial \alpha \partial c}\right) \\
E\left(\frac{\partial^2 \log L}{\partial c \partial \theta}\right) & E\left(\frac{\partial^2 \log L}{\partial c \partial \alpha}\right) & E\left(\frac{\partial^2 \log L}{\partial c^2}\right)
\end{bmatrix}
\]
\[
E \left( \frac{\partial^2 \log L}{\partial \theta^2} \right) = -n(\alpha + c + 1) \frac{1}{\theta^2} - \frac{n}{\theta^2} \left( \frac{\theta^{\alpha+1}c!+(\alpha+c)!}{(\alpha+1)\theta^{\alpha-1}c!-(\alpha+1)\theta^{\alpha}c!} \right) \left( \theta^{\alpha+1}c!+(\alpha+c)! \right)^2
\]

\[
E \left( \frac{\partial^2 \log L}{\partial \alpha^2} \right) = n \left( \frac{\theta^{\alpha+1+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right) \left( \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right)^2
\]

\[
E \left( \frac{\partial^2 \log L}{\partial c^2} \right) = n \left( \frac{\theta^{\alpha+1+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right) \left( \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right)^2
\]

and

\[
E \left( \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \right) = E \left( \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \right) = \frac{n}{\theta} \left( \frac{\theta^{\alpha+1}c!+(\alpha+c)!}{(\alpha+1)\theta^{\alpha-1}c!-(\alpha+1)\theta^{\alpha}c!} \right) \left( \frac{\theta^{\alpha+1}c!+(\alpha+c)!}{(\alpha+1)\theta^{\alpha}c!} \right)^2
\]

\[
E \left( \frac{\partial^2 \log L}{\partial \theta \partial c} \right) = E \left( \frac{\partial^2 \log L}{\partial c \partial \theta} \right) = \frac{n}{\theta} \left( \frac{\theta^{\alpha+1}c!+(\alpha+c)!}{(\alpha+1)\theta^{\alpha-1}c!-(\alpha+1)\theta^{\alpha}c!} \right) \left( \frac{\theta^{\alpha+1}c!+(\alpha+c)!}{(\alpha+1)\theta^{\alpha}c!} \right)^2
\]

\[
E \left( \frac{\partial^2 \log L}{\partial \alpha \partial c} \right) = E \left( \frac{\partial^2 \log L}{\partial c \partial \alpha} \right) = n \left( \frac{\theta^{\alpha+1+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right) \left( \frac{\theta^{\alpha+1}}{\theta^{\alpha+1}c!+(\alpha+c)!} \right)^2
\]

Since \( \gamma \) being unknown, we estimate \( I^{-1}(\gamma) \) by \( I^{-1}(\hat{\gamma}) \) and this can be used to obtain asymptotic confidence intervals for \( \theta, \alpha \) and \( c \).
X. DATA EVALUATION

In this portion, here we employed the two real lifetime data sets and analyzed them by using the weighted Shukla distribution in order to show that the weighted Shukla distribution fits better than Shukla, Shanker and Lindley distribution. The following two data sets are given below.

Data set 1: The following data set represents the breaking stress of carbon fibres of 50 mm length (GPa) reported by Nichols and Padgetcit14. The data set is given below in table 1.

Table 1: Data regarding breaking stress of carbon fibres of 50mm length (GPa) by Nichols and Padgetcit14 (2006).

<table>
<thead>
<tr>
<th>0.39</th>
<th>0.85</th>
<th>1.08</th>
<th>1.25</th>
<th>1.47</th>
<th>1.57</th>
<th>1.61</th>
<th>1.61</th>
<th>1.69</th>
<th>1.80</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.84</td>
<td>1.87</td>
<td>1.89</td>
<td>2.03</td>
<td>2.03</td>
<td>2.05</td>
<td>2.12</td>
<td>2.35</td>
<td>2.41</td>
<td>2.43</td>
</tr>
<tr>
<td>2.48</td>
<td>2.50</td>
<td>2.53</td>
<td>2.55</td>
<td>2.55</td>
<td>2.56</td>
<td>2.59</td>
<td>2.67</td>
<td>2.73</td>
<td>2.74</td>
</tr>
<tr>
<td>2.79</td>
<td>2.81</td>
<td>2.82</td>
<td>2.85</td>
<td>2.87</td>
<td>2.88</td>
<td>2.93</td>
<td>2.95</td>
<td>2.96</td>
<td>2.97</td>
</tr>
<tr>
<td>3.09</td>
<td>3.11</td>
<td>3.11</td>
<td>3.15</td>
<td>3.15</td>
<td>3.19</td>
<td>3.22</td>
<td>3.22</td>
<td>3.27</td>
<td>3.28</td>
</tr>
<tr>
<td>3.31</td>
<td>3.31</td>
<td>3.33</td>
<td>3.39</td>
<td>3.39</td>
<td>3.56</td>
<td>3.60</td>
<td>3.65</td>
<td>3.68</td>
<td>3.70</td>
</tr>
<tr>
<td>3.75</td>
<td>4.20</td>
<td>4.38</td>
<td>4.42</td>
<td>4.70</td>
<td>4.90</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data set 2: The data set reported by Smith and Naylor (1987) regarding the strength of 1.5cm glass fibres measured at the National physical laboratory England. The data set is given below in table 2

Table 2: Data reported by Smith and Naylor (1987) regarding strength of 1.5cm glass fibers at National Physical Laboratory England

<table>
<thead>
<tr>
<th>0.55</th>
<th>0.93</th>
<th>1.25</th>
<th>1.36</th>
<th>1.49</th>
<th>1.52</th>
<th>1.58</th>
<th>1.61</th>
<th>1.64</th>
<th>1.68</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.73</td>
<td>1.81</td>
<td>2</td>
<td>0.74</td>
<td>1.04</td>
<td>1.27</td>
<td>1.39</td>
<td>1.49</td>
<td>1.53</td>
<td>1.59</td>
</tr>
<tr>
<td>1.61</td>
<td>1.66</td>
<td>1.68</td>
<td>1.76</td>
<td>1.82</td>
<td>2.01</td>
<td>0.77</td>
<td>1.11</td>
<td>1.28</td>
<td>1.42</td>
</tr>
<tr>
<td>1.5</td>
<td>1.54</td>
<td>1.6</td>
<td>1.62</td>
<td>1.66</td>
<td>1.69</td>
<td>1.76</td>
<td>1.84</td>
<td>2.24</td>
<td>0.81</td>
</tr>
<tr>
<td>1.13</td>
<td>1.29</td>
<td>1.48</td>
<td>1.5</td>
<td>1.55</td>
<td>1.61</td>
<td>1.62</td>
<td>1.66</td>
<td>1.7</td>
<td>1.77</td>
</tr>
<tr>
<td>1.84</td>
<td>0.84</td>
<td>1.24</td>
<td>1.3</td>
<td>1.48</td>
<td>1.51</td>
<td>1.55</td>
<td>1.61</td>
<td>1.63</td>
<td>1.67</td>
</tr>
<tr>
<td>1.7</td>
<td>1.78</td>
<td>1.89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

R software is used to carried out the calculation of unknown parameters along with calculation of model comparison criterion values like AIC, AICC and BIC. In order to compare the weighted Shukla distribution with Shukla, Shanker and Lindley distributions. We employ the values of criterion like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion). The better distribution corresponds to lesser AIC, AICC and BIC values. The formulas for calculation of AIC, AICC and BIC are
Where \( k \) is the number of parameters and \( n \) is the sample size in the statistical model and \(-2 \log L\) is the maximized value of the log-likelihood function under the considered model. From results given in table 3 below, it has been observed that the weighted Shukla distribution have the lesser AIC, AICC, BIC and \(-2 \log L\) values as compared to the Shukla, Shanker and lindley distribution, which clearly indicates that the weighted Shukla distribution fits better over Shukla, Shanker and lindley distribution for the two data sets given. Hence we can conclude that the weighted Shukla distribution leads to a better fit than the Shukla, Shanker and lindley distribution.

**Table 3- Parameter Estimates, Criterion values (AIC, AICC, BIC) and \(-2 \log L\) and Comparison of weighted Shukla distribution with Shukla, Shanker and lindley distribution**

<table>
<thead>
<tr>
<th>Data sets</th>
<th>Distribution</th>
<th>MLE</th>
<th>S.E</th>
<th>-2LogL</th>
<th>AIC</th>
<th>BIC</th>
<th>AICC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Weighted Shukla</td>
<td>( \hat{a} = 8.1408783 )</td>
<td>( \hat{\theta} = 2.0086212 )</td>
<td>( \hat{c} = 3.1038050 )</td>
<td>171.5877</td>
<td>177.5877</td>
<td>184.1567</td>
</tr>
<tr>
<td></td>
<td>Shukla</td>
<td>( \hat{a} = 10.8221652 )</td>
<td>( \hat{\theta} = 1.9962593 )</td>
<td>( \hat{c} = 3.6584008 )</td>
<td>198.6273</td>
<td>202.6273</td>
<td>207.0066</td>
</tr>
<tr>
<td></td>
<td>Shanker</td>
<td>( \hat{\theta} = 0.6028862 )</td>
<td>( \hat{\theta} = 0.0490024 )</td>
<td>( \hat{\theta} = 0.6731624 )</td>
<td>239.4421</td>
<td>241.4421</td>
<td>243.6317</td>
</tr>
<tr>
<td></td>
<td>Lindley</td>
<td>( \hat{\theta} = 0.59025384 )</td>
<td>( \hat{\theta} = 0.65324128 )</td>
<td>( \hat{\theta} = 0.65324128 )</td>
<td>244.7681</td>
<td>246.7681</td>
<td>248.9578</td>
</tr>
<tr>
<td>2</td>
<td>Weighted Shukla</td>
<td>( \hat{a} = 0.8759322 )</td>
<td>( \hat{\theta} = 5.0330867 )</td>
<td>( \hat{c} = 11.6211959 )</td>
<td>47.89359</td>
<td>53.89359</td>
<td>60.323</td>
</tr>
<tr>
<td></td>
<td>Shukla</td>
<td>( \hat{a} = 1.0142223 )</td>
<td>( \hat{\theta} = 0.3391657 )</td>
<td>( \hat{c} = 9.603852 )</td>
<td>162.2764</td>
<td>166.2764</td>
<td>170.5626</td>
</tr>
<tr>
<td></td>
<td>Shanker</td>
<td>( \hat{\theta} = 0.9562642 )</td>
<td>( \hat{\theta} = 0.0811598 )</td>
<td>( \hat{\theta} = 0.1274533 )</td>
<td>162.2781</td>
<td>164.2781</td>
<td>166.4213</td>
</tr>
<tr>
<td></td>
<td>Lindley</td>
<td>( \hat{\theta} = 0.99611639 )</td>
<td>( \hat{\theta} = 0.09484179 )</td>
<td>( \hat{\theta} = 0.99611639 )</td>
<td>162.5569</td>
<td>164.5569</td>
<td>166.7</td>
</tr>
</tbody>
</table>

**XI. CONCLUSION**

In the present manuscript, we have studied a new generalization of Shukla distribution by using the weighted technique known as weighted Shukla distribution. The subject distribution is generated by taking the two parameter Shukla distribution as the base distribution. The different mathematical and statistical properties of the newly introduced distribution along with reliability measures have been discussed. The parameters of the newly proposed model has also been estimated by using the method of maximum likelihood estimation. Finally, the newly introduced model is also demonstrated with the application of two real lifetime data sets and

\[
\text{AIC} = 2k - 2 \log L \\
\text{AICC} = \frac{2(k + 1)}{n - k - 1} \\
\text{BIC} = k \log n - 2 \log L
\]
then after that the results of the newly proposed weighted shukla distribution are compared with the results of Shukla, Shanker and Lindley distribution and the results show that the weighted Shukla distribution leads to a quite satisfactory over Shukla, Shanker and Lindley distribution.

REFERENCES